☞ ***Propositional* clausal logic**

✓expressions that can be true or false

☞ ***Relational* clausal logic**

✓constants and variables refer to objects

☞ ***Full* clausal logic**

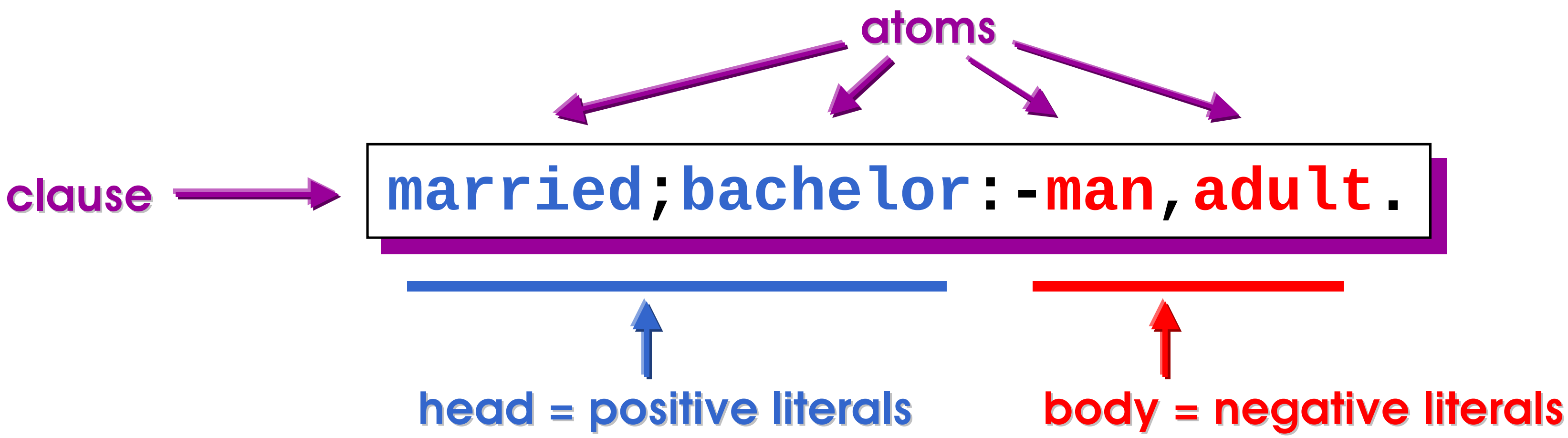
✓functors aggregate objects

☞ ***Definite clause* logic = pure Prolog**

✓no disjunctive heads

Clausal logic

"Somebody is married **or** a bachelor **if** he is a man **and** an adult."



$\texttt{married} \vee \texttt{bachelor} \vee \neg\texttt{man} \vee \neg\texttt{adult}$

Propositional clausal logic: syntax

☞ Persons are happy or sad

```
happy;sad:-person.
```

☞ No person is both happy and sad

```
:-person,happy,sad.
```

☞ Sad persons are not happy

```
:-person,sad,happy.
```

☞ Non-happy persons are sad

```
sad;happy:-person.
```

Exercise 2.1

- ***Herbrand base***: set of atoms

```
{married,bachelor,man,adult}
```

- ***Herbrand interpretation***: set of **true** atoms

```
{married,man,adult}
```

- A clause is **false** in an interpretation if all body-literals are **true** and all head-literals are **false**…

```
bachelor:-man,adult.
```

- …and **true** otherwise: the interpretation is a ***model*** of the clause.

```
:-married,bachelor.
```

Propositional clausal logic: semantics

- A clause ***C*** is a ***logical consequence*** of a program (set of clauses) ***P*** iff every model of ***P*** is a model of ***C***.

- Let ***P*** be

```
married;bachelor:-man,adult.
man.
:-bachelor.
```

- **`married:-adult`** is a logical consequence of ***P***;

- **`married:-bachelor`** is a logical consequence of ***P***;

- **`bachelor:-man`** is not a logical consequence of ***P***;

- **`bachelor:-bachelor`** is a logical consequence of ***P***.

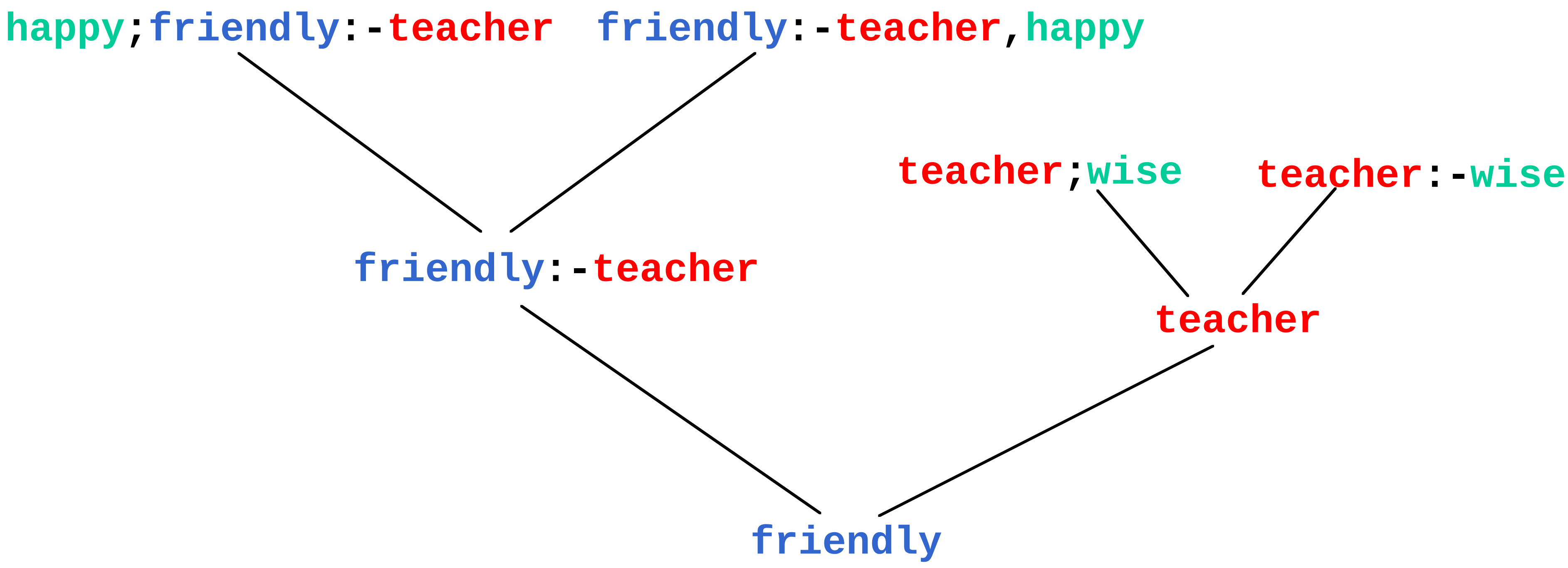
Exercise 2.2


```
has_wife:-man,married          married;bachelor:-man,adult

            has_wife;bachelor:-man,adult


square:-rectangle,equal_sides   rectangle:-parallelogram,right_angles

        square:-parallelogram,right_angles,equal_sides
```

Propositional resolution

- ☞ Propositional resolution is
  - ✓ ***sound***: it derives only logical consequences.
  - ✓ ***incomplete***: it cannot derive arbitrary tautologies like `a:-a`…
  - ✓ …but ***refutation-complete***: it derives the empty clause from any inconsistent set of clauses.

- ☞ ***Proof by refutation***: add the negation of the assumed logical consequence to the program, and prove inconsistency by deriving the empty clause.


happy;friendly:-teacher
friendly:-teacher,happy
teacher;wise
teacher:-wise
friendly:-teacher
teacher
friendly

Exercise 2.4

Direct proof:

```
friendly:-happy        happy:-has_friends
          \               /
       friendly:-has_friends
```

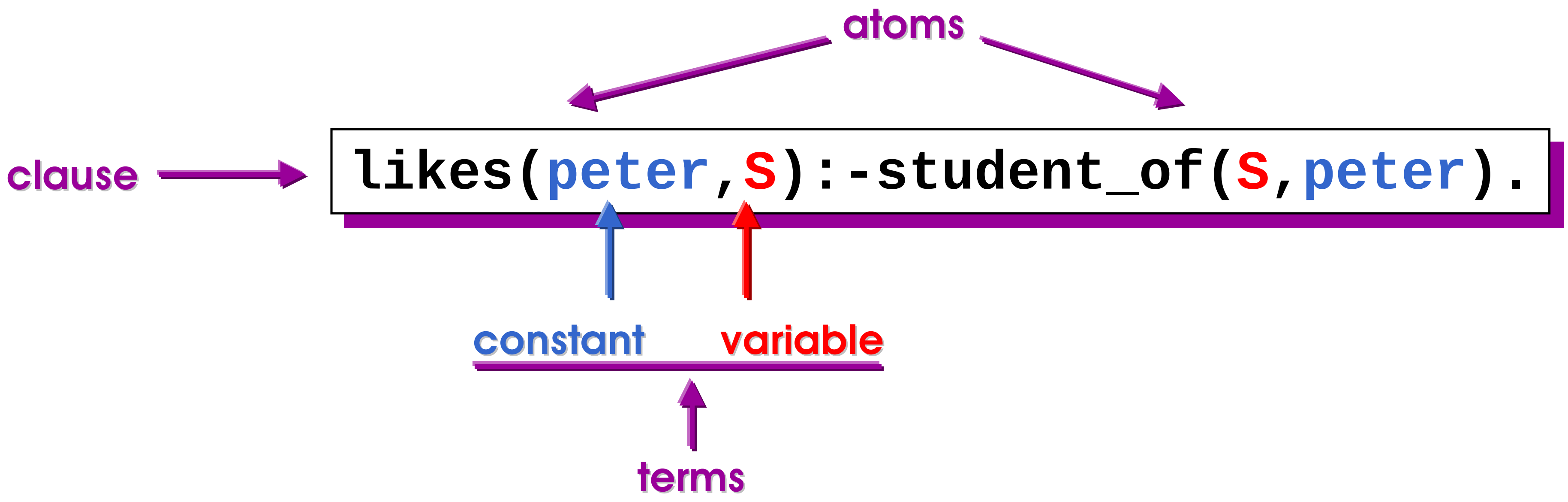
```
:-friendly        friendly:-happy
     |           /
:-happy           happy:-has_friends
     |           /
:-has_friends     has_friends
     |           /
    []
```

Proof by refutation:

$\neg$(friendly:-has_friends) $\Rightarrow$
$\neg$(friendly $\vee$ $\neg$has_friends) $\Rightarrow$
($\neg$friendly) $\wedge$ (has_friends) $\Rightarrow$
:-friendly and has_friends

Exercise 2.5

"Peter likes anybody who is his student."



Relational clausal logic: syntax

- A ***substitution*** maps variables to terms:

```
{S->maria}
```

- A substitution can be ***applied*** to a clause:

```
likes(peter,maria):-student_of(maria,peter).
```

- The resulting clause is said to be an ***instance*** of the original clause, and a ***ground instance*** if it does not contain variables.

- Each instance of a clause is among its logical consequences.

Substitutions

- ***Herbrand universe***: set of ground terms (i.e. constants)

```
{peter,maria}
```

- ***Herbrand base***: set of ground atoms

```
{likes(peter,peter),likes(peter,maria),likes(maria,peter),
 likes(maria,maria),student_of(peter,peter),student_of(peter,maria),
 student_of(maria,peter),student_of(maria,maria)}
```

- ***Herbrand interpretation***: set of **true** ground atoms

```
{likes(peter,maria),student_of(maria,peter)}
```

- An interpretation is a **model** for a clause if it makes all of its ground instances **true**

```
likes(peter,maria):-student_of(maria,peter).

likes(peter,peter):-student_of(peter,peter).
```

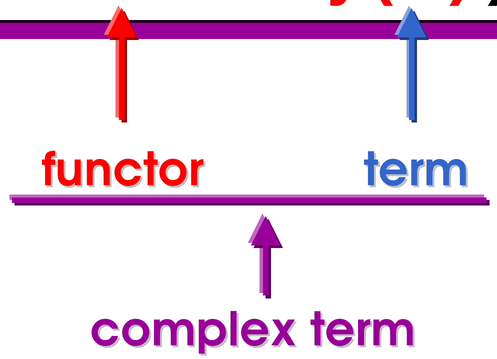
Relational clausal logic: semantics


```
:-likes(peter,N)                          likes(peter,S):-student_of(S,peter)
                                          {S->N}
:-student_of(N,peter)                     student_of(S,T):-follows(S,C),teaches(T,C)
                                          {S->N,T->peter}
:-follows(N,C),teaches(peter,C)           follows(maria,ai_techniques)
                                          {N->maria,C->ai_techniques}
:-teaches(peter,ai_techniques)            teaches(peter,ai_techniques)

[]
```

Relational resolution

"Everybody loves somebody."

```
loves(X,person_loved_by(X)).
```




```
loves(peter,person_loved_by(peter)).
loves(anna,person_loved_by(anna)).
loves(paul,person_loved_by(paul)).
…
```

Full clausal logic: syntax

☞ Every mouse has a tail

```
tail_of(tail(X),X):-mouse(X).
```

☞ Somebody loves everybody

```
loves(person_who_loves_everybody,X).
```

☞ Every two numbers have a maximum

```
maximum_of(X,Y,max(X,Y)):-number(X),number(Y).
```

Exercise 2.9

- ***Herbrand universe***: set of ground terms

```
{0,s(0),s(s(0)),s(s(s(0))),…}
```

- ***Herbrand base***: set of ground atoms

```
{plus(0,0,0), plus(s(0),0,0), …,
 plus(0,s(0),0), plus(s(0),s(0),0), …,
 …,
 plus(s(0),s(s(0)),s(s(s(0)))), …}
```

- ***Herbrand interpretation***: set of **true** ground atoms

```
{plus(0,0,0), plus(s(0),0,s(0)), plus(0,s(0),s(0))}
```

- Some programs have only infinite models

```
plus(0,X,X).
plus(s(X),Y,s(Z)):-plus(X,Y,Z).
```

Full clausal logic: semantics

```
plus(X,Y,s(Y))
```
and
```
plus(s(V),W,s(s(V)))
```
unify to
```
plus(s(V),s(V),s(s(V)))
```

```
length([X|Y],s(0))
```
and
```
length([V],V)
```
unify to
```
length([s(0)],s(0))
```

```
larger(s(s(X)),X)
```
and
```
larger(V,s(V))
```
do **not** unify (occur check!)

Exercise 2.11

	Propositional —	Relational —	Full clausal logic
Herbrand universe	—	`{a, b}` (finite)	`{a, f(a), f(f(a)),…}` (infinite)
Herbrand base	`{p, q}`	`{p(a,a), p(b,a),…}` (finite)	`{p(a,f(a)), p(f(a), f(f(a))),…}` (infinite)
clause	`p:-q.`	`p(X,Z):-q(X,Y),p(Y,Z).`	`p(X,f(X)):-q(X).`

Herbrand models	∅
	`{p}`
	`{p, q}`
		∅
		`{p(a,a)}`
		`{p(a,a), p(b,a), q(b,a)}`
		…
		(finite number of finite models)
			∅
			`{p(a,f(a)), q(a)}`
			`{p(f(a), f(f(a))), q(f(a))}`
			…
			(infinite number of finite or infinite models)

Meta-theory	sound refutation-complete decidable	sound refutation-complete decidable	sound (if unifying with occur check) refutation-complete semi-decidable

Summary

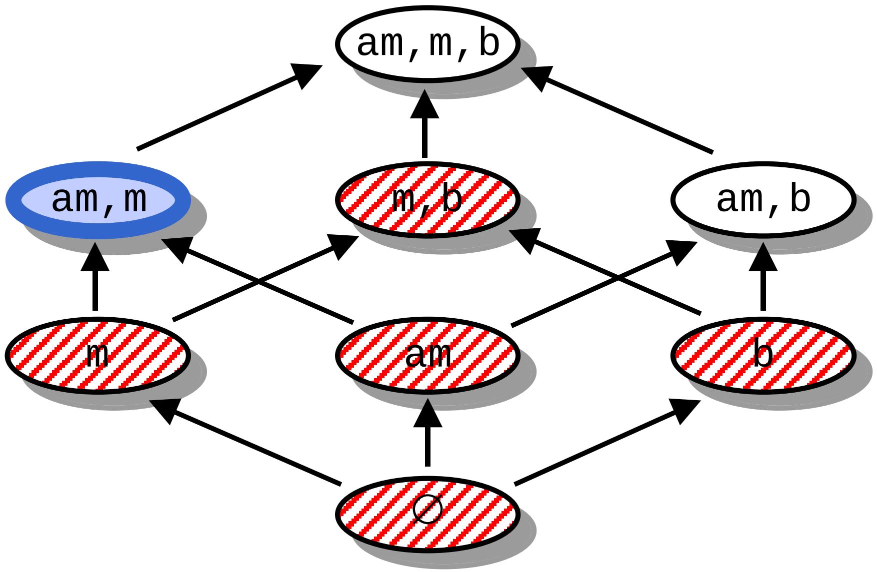

```
☞married(X);bachelor(X):-man(X),adult(X)          man(peter)
married(peter);bachelor(peter):-adult(peter)      adult(peter)
married(peter);bachelor(peter)
```

```
married(X);bachelor(X):-man(X),adult(X)           :-married(maria)
bachelor(maria):-man(maria),adult(maria)          :-bachelor(maria)
:-man(maria),adult(maria)
```

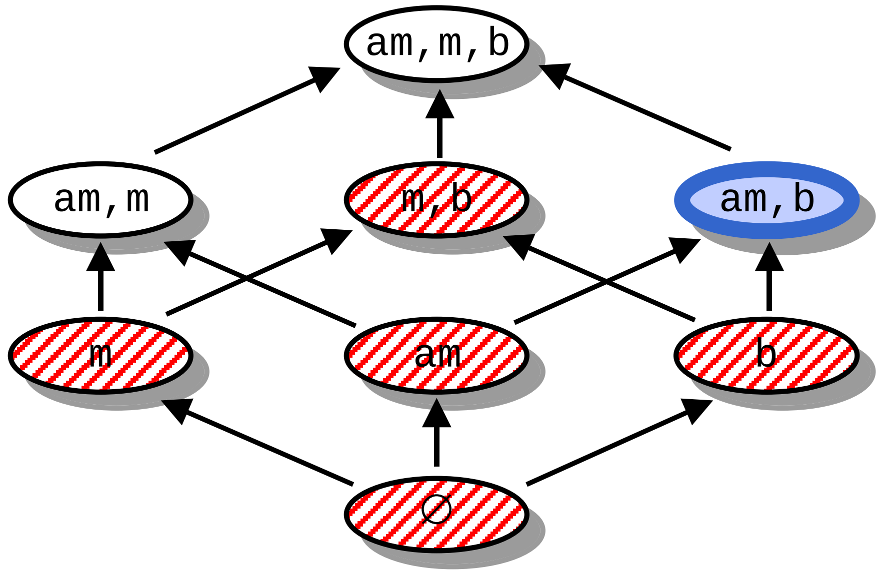
```
married(X);bachelor(X):-man(X),adult(X)           man(paul)
married(paul);bachelor(paul):-adult(paul)         :-bachelor(paul)
married(paul):-adult(paul)
```

Exercise 2.12

```
married;bachelor:-adult_man.
adult_man.
```



`married:-adult_man,not bachelor.`



`bachelor:-adult_man,not married.`

From indefinite to general clauses

- “Everyone has a mother, but not every woman has a child.”

  `∀Y∃X:mother_of(X,Y) ∧ ¬∀Z∃W:woman(Z)→mother_of(Z,W)`

- push negation inside

  `∀Y∃X:mother_of(X,Y) ∧ ∃Z∀W:woman(Z)∧¬mother_of(Z,W)`

- drop quantifiers (Skolemisation)

  `mother_of(mother(Y),Y) ∧ woman(childless_woman) ∧ ¬mother_of(childless_woman,W)`

- (convert to CNF and) rewrite as clauses

```
mother_of(mother(Y),Y).
woman(childless_woman).
:-mother_of(childless_woman,W).
```

From first-order logic to clausal logic